U.G. 3rd Semester Examination - 2021

MATHEMATICS

Course Code : BMTMCCHT302

Course Title : Algebra-II

Full Marks : 40 Time : 2 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer any **ten** questions: $1\times10=10$

a) Examine whether the operation $*$ on Q defined by $a*b = ab + 1$, is commutative.

b) Define order of an element in a group.

c) Is the symmetric group S_3 abelian? Justify.

d) Give an example of a finite non-commutative ring with unity.

e) What do you mean by divison of zero in a ring?

f) Give an example of a group (G, $*$) in which $0(a).0(b)\neq 0(a*b)$ for some $a,b\in G$.

g) Find the centraliser of (12) in the symmetric group S_3.

h) If α=(2, 5, 3)(4, 7, 8, 1) is an element of S_8, find the order of α.

i) Give an example of an infinite group, each element of which is of finite order.

j) State Lagrange's theorem.

k) Find the number of generators of the group $(\mathbb{Z}_5, +)$.

l) Define characteristic of a ring.

m) Is the dihedral group D_4 cyclic? Justify.

n) Can a non-abelian group have an abelian subgroup? Give proper reason.

o) Give an example of a finite ring R with unity 1_R and a subring S of R containing no unity.

2. Answer any **five** questions: $2\times5=10$

a) Prove that a cyclic group of prime order has no non-trivial proper subgroup.

b) Find all cyclic subgroups of the symmetric group S_3.

c) If each element in a group be its own inverse, prove that the group is abelian.

d) Write down all symmetries of a square.

e) In the symmetric group S_3, find two subgroups H and K such that $H \cup K$ is not a subgroup of S_3.

f) If G be an abelian group, show that the subset $H = \{a \in G \mid a = a^{-1}\}$ forms a subgroup of G.

g) Is the set $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + c = b + d \right\}$ a subring of the ring $M_2(\mathbb{R})$ of all 2×2 real matrices? Justify your answer.

h) If R be a ring with unity having no divisor of zero, prove that 0 and 1 are the only idempotents in R.

3. Answer any **two** questions: 5×2=10

a) i) Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$, for all $a, b \in G$.

ii) Find all elements of order 5 in the group $(\mathbb{Z}_{30}, +)$. 3+2

b) i) Prove that the characteristic of an integral domain is either zero or a prime number.

ii) Let a be a fixed element in a ring R. Define $c(a) = \{x \in R : xa = ax\}$. Prove that c(a) is a subring of R. 3+2=5

c) Prove that a finite integral domain is a field. Give an example of an infinite integral domain which is not a field. 4+1

4. Answer any **one** question: 10×1=10

a) i) Let a be an element of a group G. Prove that $o(a) = o(a^{-1})$.

ii) Let H and K be subgroups of a group G. Prove that HK is a subgroup if and only if HK=KH.

iii) If G be an abelian group of order 10 containing an element of order 5, show that G is cyclic. 3+5+2

b) i) Prove that every subgroup of a cyclic group is cyclic.

ii) Let G=<a> be a cyclic group of order 12 and H=<a^4> be a subgroup of G. Show that the distinct left cosets H in G are H, aH, a^2H and a^3H.

iii) Show that every proper subgroup of a group of order 6 is cyclic. 5+3+2

c) i) Prove that the ring $(\mathbb{Z}_n, +, .)$ is an integral domain if and only if n is a prime number.

ii) Show that in a ring R cancellation laws hold if and only if R has no divisor of zero.

iii) Find all units in the ring $(\mathbb{Z}_{10}, +, .)$.

5+3+2

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