2021
MATHEMATICS
[HONOURS]
Paper : VII

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.
Candidates are required to give their answers in their own words as far as practicable.
Notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Give frequency definition of probability.

b) Find the probability that a leap year selected at random will contain 53 Sundays.

c) What is the probability of obtaining multiple of three twice in a throw with six dice?

d) Find the constant k for which $f(x)=\int_{-\infty}^{\infty} k e^{-|t|} dt$ is a probability density function of X.

e) If X, Y be two independent random variables, then show that the correlation coefficient of X and Y is zero, provided it exists.

f) Define estimator of a population parameter. When an estimator is called unbiased estimator?

g) If the joint distribution of X and Y be given by the probability density function:

$$f(x,y)=x+y,\ 0<x<1, 0<y<1$$
$$=0,\ \text{elsewhere},$$

then find E(XY) and E(X+Y).

h) What do you mean by "weighted mean"?

i) Write down the maximum likelihood function for the normal (m,σ) population.

j) Show that the set $X=\{x:|x|\le 7\}$ is a convex set.

k) Show that the set of all feasible solutions to an L.P.P. is a closed convex set.

l) Show that the pay-off matrix given below of a game problem has no 'saddle point':

Player B

Player A	B_1	B_2	B_3
A_1	1	3	6
A_2	2	1	3
A_3	6	2	1

m) Find out the dual of the problem given by,

Minimize : $Z = 10x_1 + 2x_2$

subject to: $x_1 + 2x_2 + 2x_3 \geq 1$

$-x_1 + 2x_3 \leq 1$

and $x_1 - x_2 + 3x_3 \geq 3,\ x_1, x_2, x_3 \geq 0.$

n) Write down the dual problem of the following problem:

Maximize $Z = 8x_1 + 6x_2$

subject to $x_1 - x_2 \leq \frac{3}{5}$

$x_1 - x_2 \geq 2$

and $x_1, x_2 \geq 0$.

o) Prove that the game with pay-off matrix

Player A

Player B

	A_1	A_2	A_3
B_1	-1	2	-2
B_2	6	4	-6

, is not a fair game.

2. Answer any **three** questions: 8×3=24

a) i) If A and B be any two events corresponding to a random experiment, then show that $P(A+B) = P(A) + P(B) - P(AB)$.

ii) Let $A_1, A_2, \ldots. A_n$ be any n events connected to a random experiment E. Prove that

$$P(A_1 + A_2 + \ldots. + A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n).$$

iii) Two cards are drawn from a well-shuffled pack. Find the probability that at least one of them is spade.

3+3+2

b) i) Show that Poisson distribution is the limiting case of Binomial distribution under certain condition to be stated by you.

ii) Let 'X' be a binomially distributed random variable with parameters n and p. For what value of p is $\text{Var}(X)$ maximum, assuming that n is fixed?

iii) If 'X' be normal $(0, 1)$ distribution, find the distribution of e^{-X}. 4+2+2

c) i) The density function of a two-dimensional random variable (X, Y) is given by

$$f(x, y) = Kxy(x+y) \quad 0 \le x \le 1,\ 0 \le y \le 1;$$
$$= 0 \quad \text{elsewhere}$$

Find the value of K, the marginal density function of X and Y and the value of $P\left(\frac{1}{2} \le X \le \frac{3}{4}, \frac{1}{3} \le Y \frac{2}{3}\right)$.

ii) Find the mode/modes of the Poisson distribution. 5+3

d) i) State and prove Tchebycheff's inequality.

ii) The random variable X is normal (m, r). Find the distribution of $Y = aX + b$ where a, b are constants. (1+3)+4

e) i) If $\{X_n\}$ be a sequence of random variables such that for any n, x_n has a finite mean m_n and finite standard

deviation σ_n, then prove that

$X_n - m_n \xrightarrow[\text{in p}]{} O$ as $n \to \infty$, provided $\lim\limits_{n\to\infty} \sigma_n = 0$.

ii) Show that in 2000 throws with a coin, the probability that the number of heads lies between 900 and 1100 is at least $\frac{19}{20}$. 4+4

3. Answer any **two** questions: 8×2=16

a) i) Determine first two quartiles Q_1 and Q_2 from the following frequency distribution table:

Marks:	10-19	20-29	30-39	40-49	50-59	60-69	Total
Frequency:	8	11	15	17	17	7	75

ii) If $X_1, X_2, ..., X_n$ be a random sample from a Normal (μ, σ^2) distribution, then show that $\frac{(n-1)S^2}{\sigma^2}$ is a Chi-square distribution with $(n-1)$ degrees of freedom where S^2 is the sample variance. 4+4

b) i) Find the maximum likelihood estimator of the parameter λ of the given distribution $f(x)=\lambda\alpha x^{\alpha-1}e^{-\lambda x^{\alpha}}, x>0$ of the random variable X, assuming that α is known. (use a sample of size n).

ii) If Z has a standard normal distribution and U has a chi-square distribution with n degrees of freedom and if Z & U are independent then show that $\frac{Z\sqrt{n}}{\sqrt{U}}$ has t-distribution with n degrees of freedom. 3+5

c) i) Define correlation coefficient between the values of two variables and prove that the coefficient of correlation is independent of the change of origin and the scale of measurement.

ii) Find Skewness and Kurtosis for the following data:

Class:	0-10	10-20	20-30	30-40	Total
frequency	1	3	4	2	10

(1+3)+4

4. Answer any **five** questions: 8×5=40

a) i) Show graphically that the L.P.P

Maximize : $Z=4x_1+2x_2$

subject to

$2x_1+x_2\le 4,\ 5x_1+3x_2\ge 15,\ x_1,x_2\ge 0$

has no feasible solution.

ii) Find all basic feasible solutions of the set of equations:

$x_1+x_2+2x_3=9,\ 3x_1+2x_2+5x_3=22$

4+4

b) i) If an L.P.P.,

Optimize $Z=Cx$,

subject to $Ax=b,\ x\ge 0$;

where A is m×n coefficient matrix $(m<n)$ and rank of A is m, admits an optimal solution then show that there exists at least one basic feasible solution which will be optimal.

ii) Find all the extreme points of a closed circular region in a plane. Give reasons. 6+2

c) i) Using Charne's Big-M method, solve the LPP given by,

Maximize : $Z = 3x_1 + 2x_2$

subject to :

$x_1 + x_2 \geq 1,\ 2x_1 + x_2 \leq 4,\ 5x_1 + 8x_2 \leq 15,$
$x_1 \geq 0,\ x_2 \geq 0$

ii) State the criterion for existing unbounded solution or an infinite number of solutions to an LPP during computation of the problem by simplex method. 6+2

d) i) If an L.P.P. has at least two optimal feasible solutions then show that there exist infinite number of optimal solutions to the problem.

ii) Using Big-M method, solve the following L.P.P.:

Maximize $Z = 2x_1 + x_2 + 3x_3$

subject to $x_1 + x_2 + x_3 \leq 5$

$2x_1 + 3x_2 + 4x_3 = 12$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

2+6

e) i) State the importances of duality theory.

ii) Solving the dual problem find the optimal solutions of the primal problem by simplex method:

Maximize : $Z = 3x_1 + 9x_2 + x_3$

subject to $x_1 + 4x_2 - 2x_3 \leq 8$
$x_1 + 2x_2 \leq 4$
$x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0$

2+6

f) i) Show that the dual of the dual of an L.P.P. is the primal problem itself.

ii) Solve the following L.P.P.:

Maximize $Z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \leq 10$

$2x_1 + 3x_2 \leq 18$

$x_1 \leq 8$

$x_2 \leq 6$

$x_1 \geq 0,\ x_2 \geq 0$

by solving the dual problem. 2+6

g) i) Solve the following travelling salesman problem by giving the optimal route and the minimum cost:

To

From		A	B	C	D
	A	–	46	16	40
	B	41	–	50	40
	C	82	32	–	60
	D	40	40	36	–

ii) State and prove the necessary and sufficient conditions for the existence of a feasible solution to a transportation problem. 4+4

h) i) A college has four staff and they have to perform four tasks. The time each staff would take to perform each task is given in the matrix below:

Staff

Tasks	**E**	**F**	**G**	**H**
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a staff, so as to minimize the total working-hours?

ii) A salesman travels from one place to another; he cannot, however, travel from one place to itself. The distances (in km) between pairs of cities are given below:

To

From		**P**	**Q**	**R**	**S**
	P	–	15	25	20
	Q	22	–	45	55
	R	40	30	–	25
	S	20	26	38	–

Find out an optimum route which enables him to visit each of the cities only once, so that the total distance covered by him is minimum.

4+4

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