2021
MATHEMATICS
[HONOURS]
Paper : VI

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.
Candidates are required to give their answers in their own words as far as practicable.
Notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

   i) State D'Alembert's Principle.

   ii) Is a coordinate frame rigidly attached to the rotating earth's surface an inertial frame?

   iii) If the central force is $-\frac{\mu}{r^2}(\mu > 0)$, calculate the potential, r being the radial coordinate.

   iv) Establish the relation between the rate of change of angular momentum of a moving particle and the force acting on it.

   v) Write down the work done by a force in a moving particle round a closed curve C. What happens if the force field is conservative?

   vi) How does a rigid body differ from a deformable body?

   vii) Define 'density' of a fluid at a point. When is a fluid called non-homogeneous?

   viii) Define centre of pressure of a plane area immersed in a heavy homogeneous liquid.

   ix) Write the dimensions of

      a) Work done.

      b) Angular momentum.

   x) Write down the Newton's 2nd law of motion in vector form.

   xi) What do you mean by lines of force and surfaces of equipressure?

   xii) Define a catenary of uniform strength.

   xiii) Write down the pressure density relation for a perfect gas in

      a) isothermal change of state

      b) adiabatic change of state

xiv) What is meant by the stress component τ_{xy} at a point (x,y,z) in a continuous medium?

xv) When is the equilibrium of a heavy body resting on a fixed body said to be unstable?

xvi) Distinguish between ideal and viscous fluids.

2. Answer any **five** questions: $8\times5=40$

a) Show that the angular momentum of a system of n particles about a fixed point is equal to the angular momentum of the total mass concentrated at the centre of mass about that point plus the angular momentum of the system about its centre of mass. Hence show that the angular momentum of a rigid body of mass M about the point O, moving in two dimensions is equal to $Mvp + MK^2\dot{\theta}$. The notations involved are to be explained by you. 4+4

b) i) Obtain the inertia matrix for a homogeneous solid right circular cylinder $x^2+y^2=a^2$ of mass M bounded by the planes $Z=\pm h$ with respect to the coordinate axes.

ii) Two particles of masses m_1 and m_2 attract one another according to the law of gravitation. Show that angular momentum of m_1 relative to m_2 is a constant. 4+4

c) A uniform rod of length 2a is placed with one end in contact with a perfectly rough horizontal table, held at an inclination α to the horizontal and is allowed to fall. If θ be the inclination of the rod to the horizontal at time t, write down the equation of energy. Show that its angular velocity is $\sqrt{\frac{3g}{2a}\sin\alpha}$, when it becomes horizontal. Show also that the end of the rod will not leave the plane.

d) A uniform rod of length 2a is placed with one end in contact with a smooth horizontal table and is then allowed to fall. If α be the initial inclination of the rod to the vertical, show that its angular velocity, when it is inclined at an angle θ is $\sqrt{\left[\frac{6g}{a}+\frac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}\right]}$, g being the constant acceleration due to gravity. Find also the reaction of the table.

e) A circular hoop of radius 'a', rotating in a vertical plane with spin ω and with centre at rest, is in contact with a rough plane inclined at an angle α, the angle of friction for the surfaces in contact also being α. Show that if the initial slip is down the plane, the hoop remains stationary for a time $\frac{a\omega}{g\sin\alpha}$ and it rolls down the plane with an acceleration $\frac{1}{2}g\sin\alpha$.

f) A uniform solid cylinder is held at rest with its axis horizontal on a plane whose inclination to the horizon is α. If it be then released to roll down a length L of the slope, show that the velocity of its centre of mass is then

$$\sqrt{\frac{4}{3}gL\sin\alpha}. \qquad 8$$

g) If A, B, C be the moments of inertia of a body about principal axes through the centre

of mass of the body, show that the momental ellipsoid of the body at a point (ξ, η, ζ) is given by

$$\left(\frac{A}{M}+\eta^2+\zeta^2\right)^{x^2}+\left(\frac{B}{M}+\zeta^2+\xi^2\right)^{y^2}+\left(\frac{C}{M}+\xi^2+\eta^2\right)^{z^2}$$

$- \; 2\eta\zeta yz - 2\zeta\xi zx - 2\xi\eta xy = \text{constant},$ where M is the mass of the body.

h) A uniform rod is held at an inclination α to the horizon with one end in contact with a horizontal table whose coefficient of friction is μ. If it be then released, show that it will commence to slide if

$$\mu < \frac{3\tan\alpha}{1+4\tan^2\alpha}. \qquad 8$$

3. Answer any **two** questions: 8×2=16

a) i) If $\vec{R}$ be the resultant force and $\vec{G}$ be the moment of the resultant couple of a system of forces acting on a rigid body, show that $\vec{R}\cdot\vec{R}$ and $\vec{R}\cdot\vec{G}$ are invariant.

ii) State four forces which will not appear in the equation of virtual work. 6+2

b) i) Establish the general cartesian equations of equilibrium of a string under coplanar forces.

ii) A solid circular, cylinder of radius a and height H has one end in the shape of a hemisphere. Show that it will be in stable equilibrium when standing on that end on a smooth horizontal plane, if $H < a/\sqrt{2}$. 4+4

c) i) A uniform string rests on the upper half of a smooth vertical circle, the ends reaching to the horizontal diameter. Prove that the pressure at the highest point is 2w, where w is the weight per unit length of the string.

ii) Forces P, Q, R act along y=1, z=–1; z=1, x =–1; x=1, y=–1 respectively. If the wrench reduces to a single resultant force, show that line of action of the forces must lie on the surface

$$(x-1)(y-1)(z-1)=(x+1)(y+1)(z+1).$$

4+4

4. Answer any **three** questions: 8×3=24

a) i) Show that the necessary condition of equilibrium of a fluid under external force $\vec{F}$ per unit mass is $\vec{F}\cdot \text{curl}\, F = O$.

ii) A given volume V of a liquid is acted upon by force components –x, –y, –z. Show that the equation of the free surface is $\left(x^2+y^2+z^2\right)^3 = 9V^2$. 4+4

b) i) Show that the thrust on a uniform plane area completely immersed in a heavy homogeneous liquid is equal to the product of the pressure at its C.G. and its total area.

ii) ABC is a triangular lamina, immersed in water with C in the surface and the sides AC, BC are equally inclined to the surface. Prove that the vertical through C divides the triangle into two others, the thrust on which are in the ratio $b^3+3ab^2 : a^3+3a^2b$. 4+4

c) i) A quadrant OAB of a circle of radius a is just completely immersed in a liquid with one bounding radius OA in the surface and the line OB is vertical. Show that $\left(\frac{3a}{8}, \frac{3\pi a}{16}\right)$ are the coordinates of its centre of pressure referred to OA and OB as coordinate axes.

ii) A conical vessel of height h and vertical angle 2α, contains water whose volume is one-half that of the cone. If the vessel and the contained water revolve with uniform angular velocity ω, and no water overflows, show that ω must not be greater than $\sqrt{\frac{2g}{3h}}\cot\alpha$. 4+4

d) i) A rod of small cross-section and of density ρ has a small protion of metal of weight $\frac{1}{n}$th that of the rod attached to one extremity. Prove that the rod will float at any inclination in a liquid of density σ, if $(n+1)^2\rho = n^2\sigma$.

ii) A cone whose vertical angle 2α has its lowest generator horizontal and is

filled with liquid. Prove that the resultant thrust on the curved surface is $\sqrt{1+15\sin^2\alpha}$. 4+4

e) i) If the absolute temperature T diminishes upwards in the atmosphere according to the law $T = \frac{T_0}{1+az}$, where a>0 is a constant and T is the absolute temperature at sea-level. Show that the pressure p at a height z above sea-level is given by

$$p = p_0 e^{-\frac{z}{H}\left(1+\frac{az}{2}\right)}$$

H being the height of the homogeneous atmosphere and p_0 being the pressure at sea level.

ii) In an isothermal state, a gas is acted upon by forces $-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0$. Show that density varies as $e^{\theta/K}$, where K is a constant and $\tan\theta = \frac{y}{x}$. 4+4

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