2020
MATHEMATICS
[HONOURS]
Paper : IV

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) A linear homogeneous differential equation of order two cannot have more than two solutions. Explain.

b) If $x\frac{\partial z}{\partial x}=y\frac{\partial z}{\partial y}$, find z.

c) Find the family of curves orthogonal to the family of curves given by $y=cx^2+4cx+2c$, c being an arbitrary constant.

d) Find the Wronskian of a pair of linearly independent solutions of $\frac{d^2y}{dx^2}+9y=0$.

e) Determine the solution of the system of equations $\frac{dx}{2}=\frac{dy}{3}=\frac{dz}{4}$. What does the solution represent?

f) Find the value of $L(\sin t*\cos t)$ where '$*$' denotes the convolution operation.

g) Test for integrability of the Pfaffian differential equation

$$ydx-xdy+(x+y+z)dz=0.$$

h) Explain briefly the relationship between the surface represented by $Pp+Qq=R$ and $Pdx+Qdy+Rdz=0$, where P, Q, R are functions of x and y.

i) When would you say the function $\mu(x, y)$ an integrating factor of the differential equation $M(x, y)dx+N(x, y)dy=0$? Determine the condition to be satisfied by $\mu(x, y)$.

j) Explain the concept of "terminal velocity".

k) A particle is moving along a path $r=a\sin\theta$ with constant areal velocity $(A/8)$, starting from pole. Find the time to reach the point $\theta=\theta_1$.

l) A particle describes the curve $r = ae^{\theta}$ with constant angular velocity. Show that the radial acceleration is zero.

m) Write down the relation between external force $\vec{F}$ and the potential energy V in a conservative force field. Is this field irrotational?

n) Two particles of equal mass collide while moving in opposite directions with speeds v_1 and v_2 respectively. Find their velocities just after collision which is perfectly elastic.

o) The equation of motion of a particle is $\frac{d^2x}{dt^2} = -\mu^2 x, \mu$ is a constant. Find its velocity in terms of x, given that its velocity is zero at x = a.

p) Give an example of a force field, directed towards a fixed point, which is not a central force field. Explain your answer.

MODULE–VII

(Marks : 40)

Group–A

(Ordinary Differential Equations)

2. Answer any **three** questions: 8×3=24

a) Knowing that y = x is a solution of the equation $x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0$ reduce the equation

$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = x^3$ to a first order and of first degree differential equation, and find its complete solution.

5+3

b) Given that $y = e^{2x}$ is a solution of

$$(2x+1)\frac{d^2y}{dx^2} - 4(x+1)\frac{dy}{dx} + 4y = 0.$$

Find a solution linearly independent to $y = e^{2x}$ by reducing the order. Also obtain the general solution. 6+2

c) By the method of variation of parameters obtain the solution of

$$y''(x) - 2y'(x) + y(x) = e^x \sin^2 x.$$ 8

d) Considering $\frac{dy}{dx} = p$, convert the second order differential equation

$$2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 - 3x\left(\frac{dy}{dx}\right)^2 - 3x^2\frac{dy}{dx} - x^3 + 2 = 0,$$

$x \in \mathbb{R}$, into a first order differential equation in p and x. Find the value of p. Hence solve for y. 2+3+3

e) i) Find the orthogonal trajectories of the family of cardioides $r = a(1+\cos\theta)$, $a(\in \mathbb{R})$: parameter.

ii) By reducing to normal form, solve the differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}.$$ 4+4

Group–B

(Partial Differential Equations)

3. Answer any **one** question: 8×1=8

a) Show that a necessary condition for the integrability of the differential equation $\vec{F}\cdot d\vec{r} \equiv Pdx + Qdy + Rdz = 0$ is

$$\vec{F}\cdot(\vec{\nabla}\times\vec{F}) = 0 \quad \text{where } \vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

8

b) Find the general integral of the linear P.D.E.

$$p\cos(x+y) + q\sin(x+y) = z.$$ 8

Group–C

(Laplace Transform)

4. Answer any **one** question: 8×1=8

a) Use Laplace transform technique to solve $y''(t) + k^2y(t) = f(t)$ subject to the conditions $y(0) = 0 = y'(0)$. 8

b) If $L[f(t)] = F(s)$ then show that

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}.$$ 8

MODULE–VIII

(Marks : 40)

Group–D

(Dynamics of Particle)

5. Answer any **five** questions: $8\times5=40$

a) A particle describes an ellipse of eccentricity e about a centre of force at a focus. Prove with usual notations, that $v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$, $h^2=\mu a\left(1-e^2\right)$. When the particle is at one end of a minor axis, if its velocity is found to be doubled, then prove that the new path is a hyperbola of eccentricity $\sqrt{9-8e^2}$. 4+4

b) Two perfectly inelastic bodies of masses m_1 and m_2 moving with velocities u_1 and u_2 in the same direction impinge directly. Write down the equations of motion and show that the loss of kinetic energy due to impact is $\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(u_1-u_2\right)^2$. 8

c) i) Deduce expressions for the radial and cross-radial components of acceleration in polar co-ordinates (r, θ) of the particle which describes a plane curve.

ii) If the angular velocity about the origin be a constant ω, deduce the cross-radial component of rate of change of acceleration of the particle and show that if this rate of change of acceleration is zero, then $\frac{d^2r}{dt^2}=\frac{1}{3}\omega^2 r$. 4+4

d) A particle moves in a central force field. Prove that (i) the path must be a plane curve and (ii) the angular momentum $\vec{\Omega}=\vec{r}\times\vec{p}$ is a constant of motion. 6+2

e) A particle of unit mass is projected with a velocity v at an angle α above the horizon in a medium whose resistance is k times the velocity of the particle. Show that the direction of its velocity will make an angle $\frac{\alpha}{2}$ above the horizon after a time T given by $T=\frac{1}{k}\log\left(1+\frac{kv}{g}\tan\frac{\alpha}{2}\right)$. 8

f) Define an 'apse' for a central orbit and show that at an apse the particle is moving at right angles to the radius vector.

A particle, acted on by a central attractive force $\frac{\mu}{r^3}$, is projected with a velocity $\frac{\sqrt{\mu}}{a}$ at an angle $\frac{\pi}{4}$ with its initial position vector of magnitude "a" taking the centre of force as origin. Show that the orbit is the equiangular spiral.

g) A particle moves in a straight line with acceleration directed towards a fixed point on the straight line and is inversely proportional to the square of its distance from the fixed point O. Write down its equation of motion. If the particle starts from rest from a point A at a distance 2a from O, obtain the time to cross the distance a. Also calculate the loss of potential energy to traverse the distance. 8

h) i) A particle describe a plane curve under the action of a central force P per unit mass. Prove that, in usual notation, the differential equation of the path of the particle is $\frac{h^2}{p^3}\cdot\frac{dp}{dr} = P$. 3

ii) Find the law of force to the pole when the path is the cardioid $r = a(1-\cos\theta)$ and prove that if F be the force at the apse and v be the velocity there, then $3v^2 = 4aF$. 5
