2020
MATHEMATICS
[HONOURS]
Paper : III

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Prove that a group G in which $a^2 = e$ for every element 'a' in G is Abelian.

b) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$. Find the smallest positive integer k such that $\alpha^k = e$ in S_4.

c) In a group $(G, \circ)$ 'a' is an element of order 30. Find the order of 'a^{18}'.

d) Find all cyclic subgroups of the group $(S, \bullet)$ where $S = \{1, i, -1, -i\}$.

e) Show that $3^{2n} \equiv 1 \pmod 8$, for all integers, $n \geq 1$.

f) Let $f : R \to S$ be a ring homomorphism. Show that f is injective iff $\ker f = \{0\}$.

g) Find two integers u and v satisfying 54u+24v=30.

h) Show that the product of three consecutive integers is divisible by $\lfloor 3$.

i) Solve:
$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$
$$a_{31}x + a_{32}y + a_{33}z = 1$$
where $(a_{ij})_{3\times 3}$ is an orthogonal matrix.

j) Examine whether in $\mathbb{R}^3$, the vector $(1, 0, 7)$ is in the span of $S = \{(0, -1, 2), (1, 2, 3)\}$.

k) Show that if U is a unitary matrix then U^{-1} is unitary.

l) If λ is an eigenvalue of a real orthogonal matrix A, prove that $\frac{1}{\lambda}$ is also an eigenvalue of A.

m) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(a, b) = (b, a) \forall (a, b) \in \mathbb{R}^2$. Find the eigen values of T.

n) Evaluate $\left(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j\right) a_{ik}$ where a_{ij} is skew-symmetric tensor of order 2.

o) Define covariant and contravariant tensors of order two.

p) Show that in the V_4 with line element

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2,$$

the vector $\left(1, 0, 0, \frac{\sqrt{2}}{c}\right)$ is a unit vector.

MODULE–V

(Marks : 40)

GROUP–A

Answer any **three** questions: 8×3=24

2. a) If n is a positive integer such that $n \geq 3$, then show that the symmetric group S_n is a non-commutative group.

   b) Prove that every cyclic group is abelian.

   c) Show that the order of each element in a finite group G is a divisor of G. 3+2+3

3. a) If H be a subgroup of a cyclic group G, then show that the quotient group G/H is cyclic.

   b) Let G be a group and $a \in G$. Prove that $\langle a \rangle$ is a normal subgroup of $c(a)$, the centraliser of a. 5+3

4. a) The characteristic of an integral domain is either zero or a prime number.

   b) Show that a finite integral domain is a field. 4+4

5. a) Let $(G, \circ)$ and $(G', *)$ be two groups and $\phi : G \to G'$ be an isomorphism, then prove that

      i) $o(a) = o(\phi(a))$ for every $a \in G$.

      ii) the sets G and G' have the same cardinality.

   b) Show that the groups $(Q, +)$ and $(Q^+, \bullet)$ are not isomorphic, although the sets Q and Q^+ have the same cardinality. (3+3)+2

6. a) Let G be a group and $g \in G$. Show that the mapping $I_g : G \to G$ defined by $I_g(x) = g x g^{-1}$ for all $x \in G$ is an automorphism of G.

b) Give an example of a finite ring R with unity and a subring S of R such that S contains no unity.

c) Show that the subset $S=\left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in R\right\}$ of the ring $M_2(R)$ is not an ideal of $M_2(R)$. 4+2+2

GROUP–B

Answer any **two** questions: 8×2=16

7. a) State and prove fundamental theorem of arithmetic.

b) If $2^n - 1$ be a prime, prove that n is a prime. (1+4)+3

8. a) State and prove Chinese remainder theorem.

b) If $a|b$ and $a|c$ then prove that $a|(ax+by)$, for arbitrary integers x and y. Utilise this result, show that $\gcd(a, a+2)=1$ or 2 for every integer a. (1+3)+(2+2)

9. a) State and prove Fermat's Theorem.

b) Show that $a^{12}-b^{12}$ is divisible by 91 if a and b are both prime to 91. 5+3

MODULE–VI

(Marks : 40)

GROUP–A

Answer any **four** questions: 8×4=32

10. a) Prove that

$$\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1a_2a_3a_4\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right)$$

b) Solve the system of equations by Cramar's rule:

$$x+2y-3z=1$$
$$2x-y+z=4$$
$$x+3y \quad =5$$

4+4

11. a) Prove that if A be an invertible matrix, then A^t is invertible and $\left(A^t\right)^{-1}=\left(A^{-1}\right)^t$.

b) Prove that the matrix

$$\frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$$

is orthogonal. Utilize this to solve the equations

$$x - 2y + 2z = 2$$

$$2x - y - 2z = 1$$

$$2x + 2y + z = 7.$$ 3+5

12. a) Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 0 & 0 & 5 & 8 \\ 3 & 6 & 6 & 3 \end{pmatrix}.$$

b) Show that the eigenvalues of a diagonal matrix are its diagonal elements. 4+4

13. a) Let $T: V \to W$ be a linear mapping. Show that T is injective if and only if $\text{Ker}T = \{\theta\}$.

b) Let $T: V \to W$ be a linear mapping and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V. Show that $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ generates ImT.

c) If V and W are both finite dimensional vector spaces of the same dimension then show that T is one-to-one $\Leftrightarrow$ T is onto. 3+2+3

14. a) Use Cayley-Hamilton theorem to find A^{100} where $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

b) Show that the eigenvalues of a Hermitian matrix are all real.

c) If P is a real orthogonal matrix with det $P = -1$, then prove that -1 is an eigenvalue of P. 3+3+2

15. a) Find a basis for the vector space $\mathbb{R}^3$ that contains the vector $(1, 2, 3), (3, 5, 2)$.

b) In a vector space V over a field F, for $c \in F$, $\alpha \in V$, prove that $c\alpha = \theta \Rightarrow$ either $c = 0$ or $\alpha = \theta$.

c) If λ be an eigenvalue of an $n \times n$ matrix A, prove that λ^2 is an eigenvalue of A^2. 3+3+2

16. a) Show that if an n×n matrix A over a field F has n linearly independent eigenvectors, then A is diagonalizable.

b) Reduce the quadratic form

$$2x^2 + 5y^2 + 10z^2 + 4xy + 12yz + 6zx$$

to its normal form. Find its rank and signature. 4+4

GROUP–B

Answer any **one** question: 8×1=8

17. a) Show that

$$[i\,j, k] + [k\,j, i] = \frac{\partial g_{ik}}{\partial x^j}.$$

b) The components of a contravariant vector in x-coordinate system are 8 and 4. Find the components in $\bar{x}$-coordinate system if $\bar{x}^1 = 3x^1$ and $\bar{x}^2 = 5x^1 + 3x^2$.

c) Show that in an n-dimensional space a symmetric covariant tensor of second order has at most $\frac{n(n+1)}{2}$ different components.

2+3+3

18. a) Show that the coefficients g_{ij} given by

$$(ds)^2 = g_{ij}dx^i dx^j$$

are the components of a symmetric tensor of type $(0, 2)$.

b) Show that a vector with components $\left(-1, 0, 0, \frac{1}{c^2}\right)$, c being constant, in a space with the element

$$(ds)^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^4(dx^4)^2$$

is a null vector.

c) Write down the law of transformation of the mixed tensor A^{ij}_{klm}. 4+2+2